A family of structured set covering problems

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For a 0-1 matrix $C = c_{ij}$, column j is said to cover row i if c_{ij}=1. A cover is a subset of columns covering all rows. Unweighted SET COVER (USC) is the problem of finding a cover containing as few columns as possible.

An open optimisation problem was published in a Danish journal in 2003. Not much reflection is needed to realize that the solution of a highly structured instance C of USC will answer the question posed. The instance C or rather, the family of instances $C(k)$, is a series of square matrices of size $3^k, k$=1,2, ... Provided that $C(1)$ is known, the fractal-like structure of the matrices allows for $C(k)$ to be constructed recursively for any value of k.

Let $r(k)$ be the number of columns in an optimal solution to $C(k)$. Some preliminary investigations have enabled us to determine $r(k)$, $k = 1, 2, 3, 4$ and to show that $r(5)$ must be either 12 or 13.

12 or 13? Unfortunately LP-based bounds take us nowhere in this case since the optimum is flat as a pancake. It offers some consolation though that experiments with CPLEX were not too encouraging either. For $k = 5$, $C(5)$ is a matrix of size 243x243. Nothing was returned after 24 hours CPU time. Eventually, upon an investment of 72 hours CPU time, CPLEX managed to come up with $r(5) = 12$.

The original problem asks for $r(12)$, that is, an optimal solution to a square matrix of size 531,441. After a week the originator of the problem, using an invalid argument, announced his own answer, $r(12) = 512$, and cashed the award. We have so far shown that $r(12)$ is bounded to belong to the interval [210, 377]. We also need to conclude that no brute force approach (such as further experiments with CPLEX) is likely to work whereas paper and pencil may suffice in providing the final result.

Reference

J. Krarup, J. Villadsen, "A family of structured set covering problems solvable in polynomial time", in preparation.